***NEW POINTS ON THE GRAPH yx = xy***

Thanks to Marcelo Arruda from BRAZIL for the following ideas.

Using the basic form of De Moivre’s theorem: if $z=rcisθ=\left(\cos(θ)+i\sin(θ)\right) $ then:$ z^{n}=r^{n}cisnθ=r^{n}\left(\cos(nθ)+i\sin(nθ)\right)$, so if *z* is to be REAL then

 $\sin(nθ)$ must be zero so ***θ = nπ*** (ie multiples of π rads or 1800)

Therefore if $z$ is to be real then $z^{n}=r^{n}\left(\cos(nπ)+i\sin(nπ)\right)$

Then, considering two **negative** real numbers “***a”*** and “***b”***, (N.B. the modulus is always positive so the **modulus** of “***a****”* is “**– *a”***) then we can write:

 $ z^{n}= r^{n} \left(\cos(nθ)+i\sin(nθ)\right)$

$$a^{b}=(-a)^{b}\left(\cos(bπ)+i\sin(bπ)\right)=\left(\left(-a\right)^{-b}\right)^{-1}\left(\cos(bπ)+i\sin(bπ)\right)$$

$$b^{a}=(-b)^{a}\left(\cos(aπ)+i\sin(aπ)\right)=\left(\left(-b\right)^{-a}\right)^{-1}\left(\cos(aπ)+i\sin(aπ)\right)$$

 Now we will look at the two parts of these expressions and analyse them individually:

$$a^{b}=\left(\left(-a\right)^{-b}\right)^{-1}\left(\cos(bπ)+i\sin(bπ)\right)$$

$$b^{a}=\left(\left(-b\right)^{-a}\right)^{-1}\left(\cos(aπ)+i\sin(aπ)\right)$$

So, if the **positive** numbers $x=-a$ and $y=-b$ satisfy $x^{y}=y^{x}$, then the **red** parts of above equations show this result and must be equal to each other.

(ie The equation: $x^{y}=y^{x} $becomes $\left(-a\right)^{-b}=\left(-b\right)^{-a}$)

The **blue** parts will be equal to each other if:

$\cos(bπ)=\cos(aπ)$ and $\sin(bπ)=\sin(aπ)$, which means $bπ=aπ\pm 2kπ$ and therefore $b=a\pm 2k$ .(where $k$ is any whole number)

Recall ***a*** and ***b*** are negative so multiplying that last equality by *-1* we get $-b=-a\pm 2k$ (remember “***– b***” and “***– a***” are positive numbers!)

So, if we can find pairs of **positive numbers** ***x*** and ***y*** which differ by $2k$ and which obey $x^{y}=y^{x},$ then their **opposite negative numbers**  $-x and-y$ will also satisfy the equation $(-x)^{(-y)}=(-y)^{(-x)}$.

The simplest example of this is when ***x*** = 4 and ***y*** = 2. These numbers differ by 2 and they satisfy 42 = 24 so this means that the **opposites** ***x*** = – 4 and ***y*** = – 2 will also satisfy the equation: $x^{y}=y^{x} $**because** $(-4)^{(-2)}=(-2)^{(-4)}$

To find such numbers, we let $y=x-2k$

and solve $x^{x-2k}=\left(x-2k\right)^{x}$ -------------------EQU 1

for $k= 1, 2, 3$ and so on.

**Examples:**

If **k = 1**, Equ. 1 becomes $ x^{x-2}=\left(x-2\right)^{x}$, whose solution is $x=4$.

(Found by drawing the graphs $f(x)=x^{x-2}and f\left(x\right)=\left(x-2\right)^{x} $using the AUTOGRAPH program and finding the intersection point.)

This leads to $x=4 and y=x-2=2.$ The positive solutions are **+4** and **+2**

and therefore, $-4$ and $-2$ will also satisfy $x^{y}=y^{x}$

(In each case, the ***x*** and ***y*** values can be swapped to produce $x=-2 , y=-4$)

(We already knew these solutions.)

**Now, let’s explore some new solutions using**$ y=x-2k$**:**

If **k = 2** (so the ***x*** and ***y*** differ by 4) then Equ 1 becomes$ x^{x-4}=\left(x-4\right)^{x}$ whose solution is $x=5.6647143$ (from Autograph)

This leads to $y=x-4=1.6647143$,

so $x=-5.6647143$ and $y=-1.6647143$ will be solutions too.

Testing: $\left(-5.6647143\right)^{-1.6647143}=0.0275738+0.048443i$

$$ \left(-1.6647143\right)^{-5.6647143}=0.0275738+0.048443i$$

(Again we can say $x=-1.6647143$and $y=-5.6647143$are solutions)

If **k = 3** (so the ***x*** and ***y*** differ by 6) then Equ 1 becomes $x^{x-6}=\left(x-6\right)^{x}$, whose solution is $x=7.4941717$

This leads to $y=x-6=1.4941717$,

so $x=-7.4941717$ and $y=-1.4941717$ will be solutions too.

Testing: $\left(-7.4941717\right)^{-1.4941717}=-0.000903+0.049311i$

$$ \left(-1.4941717\right)^{-7.4941717}=-0.000903+0.049311i$$

(Again $x=-1.4941717$and$y=-7.4941717$are solutions too.)

If **k = 4** (so the ***x*** and ***y*** differ by 8) then Equ 1 becomes$ x^{x-8}=\left(x-8\right)^{x}$, whose solution is $x=9.3944668$

This leads to $ y=x-8=1.3944668$,

so $x=-9.3944668$ and $y=-1.3944668$will be solutions too.

Testing: $\left(-9.3944668\right)^{-1.3944668}=-0.014319+0.041595i$

$$ \left(-1.3944668\right)^{-9.3944668}=-0.014319+0.041595i$$

(Again $x=-1.3944668$ and $y=-9.3944668$are solutionstoo**.)**

If **k = 5** (so the ***x*** and ***y*** differ by 10) then Equ 1 becomes$ x^{x-10}=\left(x-10\right)^{x}$, whose solution is $x=11.33$

This leads to $y=x-10=1.33$

so $x=-11.33$ and $y=-1.33$will be solutions too.

Testing: $(-11.33)^{-1.33}=-0.020+0.0340i$

$$ \left( -1.33\right)^{-11.33}=-0.020+0.0340i$$

(Again $x=-1.33$and $y=-11.33$are solutions too.)

We can continue this as far as we like, but the pattern is better seen graphically.

The **intersection points** ( ) of the “*hyperbola-like*” curve in the 1st quadrant, of positive ***x and y*** solutions of ***xy = yx***, with the lines ***y = x, y = x ± 2, y = x ± 4,***

***y = x ± 6*** etc., **are reflected in the line *y = – x* so that they re-appear in the 3rd quadrant but with the negative versions of the coordinates.**

We already knew the points (–4, –2), (–2.718, –2.718) and (–2, –4).

The solutions to$ x^{y}=y^{x}$previously knownare **all the points on the purple line *y = x***, **all the points on the blue “hyperbola-like curve**” and all the points denoted by red dots ( )

**The LIGHT BLUE points ( ) are the new ones.**



The solutions to$ x^{y}=y^{x}$previously knownare all the points on the purple line ***y = x***, all the points on the blue “hyperbola-like curve” and all the points denoted by red dots ( )